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Hypothesis of the Fundamental Length and Quantum Electrodynamics

Il-Tong Cheon

Department of Physics, Yonsei University, Seoul, Korea

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Under the hypothesis of a fundamental length l_0, the Bopp equation is considered as a basic equation of motion instead of the Klein–Gordon equation. Assuming that the mass is a function of l_0, we derive a mass relation $m \geq (\hbar/2l_0c)$. The propagators obtained in the framework of the present theory have the same types as those with a simple cutoff. However, because of the mass relation, the tildon field with indefinite metric is always confined in the virtual state and never appears in real processes as a physical entity. Thus, our new version leads to a finite theory of quantum electrodynamics.

1. INTRODUCTION

Quantum electrodynamics (QED) is one of the most beautiful theories that we have ever had. There is no outstanding discrepancy between theory and experiment despite our pursuing the limits of the theory to higher accuracy and smaller distances (Hofstadter, 1975; Rich and Wesley, 1972) than was possible some time ago. However, the occurrence of divergences which can temporarily be avoided only by renormalization is considered to be an essential defect of the present scheme for quantum electrodynamics. The renormalization procedure seems to be tricky for the case of infinite theory which has divergences, although it is unquestionable for the finite theory. If the mass difference between a proton and a neutron were due to the electromagnetic interaction, it might be very difficult to explain this mass difference with the renormalized theory, because the self-energy of the particle which is infinite and has to be renormalized is the quantity not to be calculated in the framework of the theory.

It is mentioned (Heisenberg, 1938; March, 1936a, b; March, 1937a, b, c)

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that any future theory of elementary particles must contain a universal length besides two fundamental constants, c and h. The introduction of such a fundamental length would eliminate the divergence difficulties from relativistic quantum field theory by cutting off the high-energy part of the proper field in the same way that the high-energy black-body radiations, which give rise to the Rayleigh–Jeans (Rayleigh, 1900; Jeans, 1905) divergence, are cut off by means of the constant h in Planck's quantum theory (Planck, 1900).

In this situation a series of discussions were given by several people on the generalized equation for propagation of the electromagnetic field (Bopp, 1940; Podolsky, 1942; Podolsky and Kikuchi, 1944; Montgomery, 1947; Green, 1947). Since this generalized equation (referred to as the Bopp equation hereafter) containing a constant which has a dimension of length gives a propagator that is superposition of a photon and a tildon, there is a possibility for removing divergences from the theory. However, as the tildon field obeys an indefinite metric, we have a positive or negative probability for even or odd numbers of tildons. Such a concept is physically unacceptable (Matthews, 1949). If the tildon field could be confined in virtual states, the difficulties associated with an indefinite metric would not take place.

In this paper, by assuming that the particle mass is a function of the fundamental length l_0, we shall derive a condition for confinement of the tildon field to virtual states.

2. HYPOTHESIS OF THE FUNDAMENTAL LENGTH AND A MASS RELATION

2.1. Speculation on the Fundamental Length. It can be said through quantum electrodynamics (QED) that all arguments given in the special theory of relativity are surely correct in the microscopic world up to the order of 10^{-15} cm, because no outstanding discrepancy between theory and experiment has ever been found. However, that does not mean to rule out the possibility that a principal modification might be required for the space-time concept in the microscale smaller than 10^{-15} cm (Pauli, 1933).

Here we must remind ourselves how to measure the distance or position in the microscopic world of the order of 10^{-15} cm or smaller. In such a small region one must do it through the interaction associated with the elementary particle. Let us put a test body in the field whose quantity we intend to measure. It is clear that we cannot measure by a test body the average of a field quantity in a volume V unless the test body itself can be localized in the volume. However, the particle with a mass m cannot be localized at a point because the particle position is uncertain by $\Delta x > (\hbar/mc)$. Furthermore, as is discussed in the Appendix, a particle in the field cannot be localized in the sphere whose radius is less than $(2F)^{1/2}(\hbar/mc)$, where F is the coupling constant associated with the field.

It follows from the foregoing that the local theory presupposes implicitly the existence of arbitrarily heavy elementary particles ($m \to \infty$). In order to avoid the appearance of divergences in the theory, it seems that we should introduce an upper limit on the mass of the elementary particle. The imposition of a certain upper bound on the particle mass m_x would mean, in principle, to set a limitation upon the use of the concept of arbitrarily exact coordinates of the point in the space and upon the applicability of local theory for scales of the order of $\Delta x \sim (\hbar/m_x c)$.

Keeping this argument in mind, we shall derive an important mass relation from a basic equation of motion which contains a fundamental length.

2.2. Derivation of the Mass Relation. Let us recall Mach's principle, "the inertial mass of a body is determined by the total distribution of matter in the universe." This means that the inertial mass of the particle depends on the structure of the space-time. The structure of the space-time might be determined by the matter distribution. From these arguments, it is rather natural to consider that the particle mass has a mutual relation with the structure of the space-time.

In the theory of special relativity, the Lagrangian for a particle is generally expressed as

$$\mathscr{L} = -m_0 c^2 \sqrt{}(1 - \beta^2)^{1/2} \tag{2.1}$$

where m_0 is the particle mass and $\beta = v/c$ with the particle velocity v and the speed of light c.

Assuming that the particle mass is a function of a universal constant l_0, we replace m_0 in equation (2.1) by $\tilde{m}(l_0)$

$$\mathscr{L} = -\tilde{m}(l_0) c^2 \sqrt{}(1 - \beta^2)^{1/2} \tag{2.2}$$

where we have a condition,

$$\tilde{m}(l_0) \to m_0 \qquad \text{for} \qquad l_0 \to 0 \tag{2.3}$$

Of course, m_0 and $\tilde{m}(l_0)$ are real. With this Lagrangian, the momentum and energy of a particle are easily obtained as

$$P_j = \frac{\partial \mathscr{L}}{\partial v_j} = \frac{\tilde{m} v_j}{(1 - \beta^2)^{1/2}} \qquad (j = 1, 2, 3)$$

$$E = \mathbf{p} \cdot \mathbf{v} - \mathscr{L} = \frac{\tilde{m} c^2}{(1 - \beta^2)^{1/2}} \tag{2.4}$$

where the argument l_0 in the expression $\tilde{m}(l_0)$ was dropped for simplicity. Thus, the dispersion law yields

$$p_\mu p^\mu \equiv p_0{}^2 - \mathrm{p}^2 = \tilde{m}^2 c^2 \tag{2.5}$$

where $p_0 = E/c$. Our metric is $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$ and the representation $a_\mu b^\mu$ is the summation over all components, i.e., $a_\mu b^\mu = \sum_{\mu\nu} g_{\mu\nu} a_\mu b_\nu$. Since $\tilde{m}$ is a function of the universal constant l_0 alone, the dispersion law (2.5) is invariant under the Lorentz transformation. The explicit expression of the lowest-order dispersion law containing l_0 is

$$\left(1 - \frac{l_0{}^2}{\hbar^2} p_\mu p^\mu\right) p_\lambda p^\lambda = m_0{}^2 c^2 \tag{2.6}$$

which is a Lorentz-invariant form as well (Bopp, 1940; Podolsky, 1942; Podolsky and Kikuchi, 1944; Montgomery, 1947; Green, 1947; Pais and Uhlenbeck, 1950). For the limit $l_0 \to 0$, the equation (2.6) reduces to the usual dispersion law, which is also obtained from equation (2.5) with the condition (2.3). Appearance of the Planck constant $\hbar$ in (2.6) is for the sake of dimension and is equivalent to introduction of quantum mechanical characters into it. Substitution of (2.5) into (2.6) gives a quadratic equation of $\tilde{m}^2$,

$$\tilde{m}^4 - \left(\frac{\hbar}{l_0 c}\right)^2 \tilde{m}^2 + \left(\frac{\hbar}{l_0 c}\right)^2 m_0{}^2 = 0 \tag{2.7}$$

There are two solutions of this equation,

$$\tilde{m}_\pm{}^2 = \frac{1}{2}\left(\frac{\hbar}{l_0 c}\right)^2 \left\{1 \pm \left[1 - \left(\frac{2 l_0 m_0 c}{\hbar}\right)^2\right]^{1/2}\right\} \tag{2.8}$$

(see Figure 1). Since the mass is real, we should have a condition in equation (2.8)

$$1 - (2 l_0 m_0 c/\hbar)^2 \geqslant 0$$

Thus we obtain a very important mass relation

$$m_0 \leqslant \frac{1}{2}\left(\frac{\hbar}{l_0 c}\right) \tag{2.9}$$

which means that the physical mass cannot be larger than $(\hbar/2l_0c)$. In the limit $l_0 \to 0$, the usual concept, i.e., $0 \leqslant m_0 \leqslant \infty$, is restored just as the special theory of relativity and quantum theory reduce to the classical one in the limit $c \to \infty$ and $\hbar \to 0$, respectively. Notice that only the solution $\tilde{m}_-{}^2$ in (2.8) satisfies the condition (2.3). When $l_0 \ll (\hbar/m_0 c)$, we have

$$\tilde{m}_-{}^2 = m_0{}^2[1 + (l_0 m_0 c/\hbar)^2 + \cdots] \tag{2.10}$$

Therefore, $\tilde{m}_-{}^2 \to m_0{}^2$ for $l_0 \to 0$. On the other hand, $m_+{}^2$ is an unphysical solution by which our initial condition (2.3) is not satisfied.

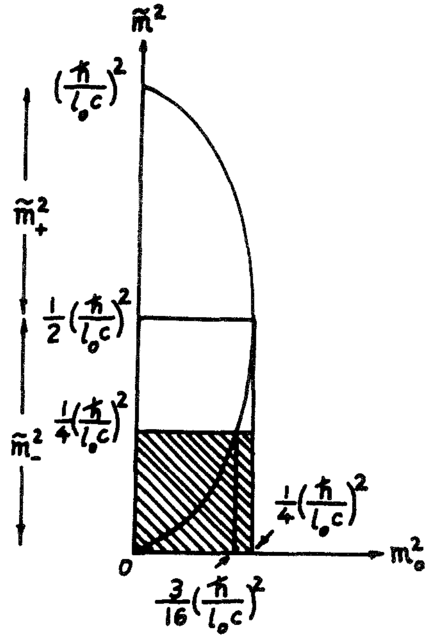

Fig. 1. The graph of $[\tilde{m}^4 - (\hbar/l_0c)^2\tilde{m}^2 + (\hbar/l_0c)^2m_0{}^2] = 0$. The physical region is shaded by lines.

When $m_0 > (\hbar/2l_0c)$, $\tilde{m}_\pm$ becomes complex, and accordingly, for this case, the particle is extremely unstable because the imaginary part becomes very large. This can easily be seen as follows: $\tilde{m}_\pm{}^2 = \frac{1}{2}(\hbar/l_0c)^2 \pm i\frac{1}{2}(\hbar/l_0c)^2a^{1/2}$ (where $a = 2m_0l_0c/\hbar > 1$) and $\text{Im}\,\tilde{m}_\pm$ must be very large since l_0 is expected to be very small. A large value of $\text{Im}\,\tilde{m}_\pm$ yields a short lifetime. Since $|\text{Im}\,\tilde{m}_\pm|$ is an order of $(\hbar/l_0c)$, the lifetime of the particle is, then, an order of $\tau \sim \hbar/[(\hbar/l_0c)c^2] = l_0/c$ by the Heisenberg's uncertainty principle. Therefore, such a particle with an extremely short life cannot be observed, because our minimum observable length is l_0.

Let us interpret the mass relation (2.9) to be generally satisfied by all physical masses of the elementary particles. Then, one finds that $\tilde{m}_-$ is physical as well as unphysical because $\tilde{m}_- \leqslant (\hbar/2^{1/2}l_0c)$, while $\tilde{m}_+$ is always unphysical because $\tilde{m}_+ \geqslant (\hbar/2^{1/2}l_0c) > (\hbar/2l_0c)$. Although the physical particle playing a role in real processes is allowed to have only a physical mass $\tilde{m}_-$, the unphysical mass $\tilde{m}_+$ would possibly appear only in virtual processes because of the mass relation. This restriction on the mass might rescue the concept of indefinite metric (Pauli, 1943; Matthews, 1949; Gupta, 1950; Feynman, 1949) from its difficult situation, such as negative probability (a brief explanation will be given in Section 3.4), and hopefully remove all divergences in quantum electrodynamics.

3. EQUATION OF MOTION

In this section, we shall discuss the basic equations of motion corresponding to the Klein–Gordon equation and the Dirac equation.

3.1. Bopp Equation. In our version, the lowest-order dispersion law containing l_0 is given in (2.6), and it can also be expressed as

$$-\frac{l_0^2}{\hbar^2}(p^2 - \tilde{m}_+^2c^2)(p^2 - \tilde{m}_-^2c^2) = 0 \tag{3.1}$$

where $p^2 \equiv p^\mu p_\mu \equiv p_\mu p^\mu$. By replacing the momentum in (2.6) or (3.1) by the differential operators in the usual way,

$$\begin{aligned} p_\mu &\to i\hbar \frac{\partial}{\partial x^\mu} \\ \frac{\partial}{\partial x^\mu} &= \left\{\frac{\partial}{\partial(ct)}, \nabla\right\} \\ \frac{\partial}{\partial x_\mu} &= \left\{\frac{\partial}{\partial(ct)}, -\nabla\right\} \end{aligned} \tag{3.2}$$

we find the Bopp equation

$$\left(1 + l_0^2 \frac{\partial^2}{\partial x_\mu\, \partial x^\mu}\right) \frac{\partial^2}{\partial x_\lambda\, \partial x^\lambda} \psi = -\left(\frac{m_0 c}{\hbar}\right)^2 \psi \tag{3.3}$$

In the limit of $l_0 \to 0$, this equation reduces to the usual Klein–Gordon equation.

This type of equation was already discussed by several people (Bopp, 1940; Podolsky, 1942; Podolsky and Kikuchi, 1944; Montgomery, 1947; Green, 1947; Pais and Uhlenbeck, 1950) for the case of $m_0 = 0$. However, their works have never fully been accepted because the additional meson field has the indefinite metric, which is not consistent with our usual concept of probability (Matthews, 1949). In our case, the additional field obeying the indefinite metric is confined in the virtual state by the mass relation (2.9).

For the case of the massless particle, i.e., $m_0 = 0$, the Bopp equation takes the form

$$(1 + l_0^2\, \partial^\mu\, \partial_\mu)\, \partial^\lambda\, \partial_\lambda \psi = 0 \tag{3.4}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$. It is to be stressed that because of the constancy of the speed of light, the motion of the massless particle in a free space (no interaction) is described by the usual Klein–Gordon equation

$$\partial^\lambda\, \partial_\lambda \psi = 0 \tag{3.5}$$

which is also the solution of equation (3.4). The solution of equation (3.4) is given as

$$\psi(x) = \frac{1}{(2\pi)^4}\int [\psi(\mathbf{k}) \exp(ik_\mu x^\mu) + \psi^*(\mathbf{k}) \exp(-ik_\mu x^\mu)]\, d^4k$$
$$+ \frac{1}{(2\pi)^4}\int [\phi(\bar{\mathbf{k}}) \exp(i\bar{k}_\mu x^\mu) + \phi^*(\bar{\mathbf{k}}) \exp(-i\bar{k}_\mu x^\mu)]\, d^4\bar{k} \quad (3.6)$$

where

$$k_0 = |\mathbf{k}|, \qquad \bar{\mathbf{k}} = \mathbf{k}, \qquad \text{and} \qquad \bar{k}_0 = (1 + l_0^2 k_0^2)^{1/2}/l_0 = (\mathbf{k}^2 + 1/l_0^2)^{1/2}$$

The second term in (3.6) is not a physical field but only a reflection of the discrete space-time associated with the universal length l_0. Let us refer to this unphysical field as a "tildon field" (Matthews, 1949). And, then, $\psi(x)$ is the superposition of a photon and a tildon with a mass $(\hbar/l_0 c)$. This tildon field makes a negative contribution to the total energy and induces negative probability because it belongs to the indefinite metric. However, because of the mass relation (2.9), this field cannot appear in real processes as a physical entity except only in virtual processes. Therefore, for the motion of the free photon, only the free electromagnetic field can appear. Since our basic equation of motion is given by (2.1), a tildon appears together with a particle in the virtual state while only the particle exists in the real state. Therefore, we have the following relation:

$$(\text{number of the particle}) \geqslant (\text{number of the tildon})$$

3.2. Static Solution. In order to see the effects of the l_0-dependent term in equation (3.4), let us consider the static case for $m_0 = 0$,

$$\left(1 - l_0^2 \frac{\partial^2}{\partial x_j\, \partial x^j}\right) \frac{\partial^2}{\partial x_i\, \partial x^i}\phi = 0 \qquad (i, j = 1, 2, 3) \quad (3.7)$$

The solution of this equation is

$$\phi(r) = -(\epsilon/r)(1 - e^{-r/l_0}) \quad (3.8)$$

where ϵ is an arbitrary constant and $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. When $l_0 \to 0$, the solution (3.8) reduces to the usual static potential for the electric field. As is shown in Figure 2, our solution is finite at the origin, while the usual static potential is infinite there. Since the difference between the two cases is significant in the short-range region, our theory might be tested at very high energy.

With the static potential (3.8), we are able to calculate the total energy of the electric field. The expression of the total energy is

$$U = \frac{1}{8\pi}\int_0^\infty E(r)^2 4\pi r^2\, dr \quad (3.9)$$

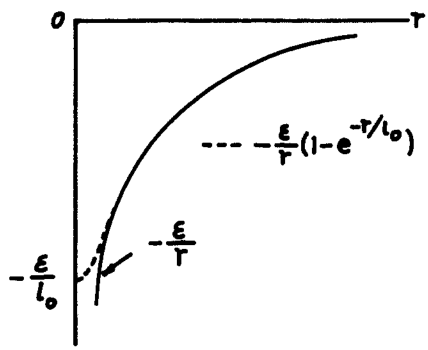

Fig. 2. The static Coulomb potential.

where the electric field is given by

$$E(r) = -\frac{\partial \phi(r)}{\partial r} \tag{3.10}$$

For the usual case, i.e., $\phi = -e/r$ (where e is the electric charge), the total energy U becomes infinite. When the charge of an electron is assumed to uniformly distribute over a sphere of radius r_0 in order to avoid such divergent results, we obtain

$$U = \frac{1}{8}\int_{r_0}^{\infty} E(r)^2 4\pi r^2\, dr = \frac{e^2}{2r_0} \tag{3.11}$$

Under the hypothesis that the electron mass is of electromagnetic origin, the classical electron radius is obtained by putting the total energy (3.11) equal to the electron rest mass energy,

$$r_0 = \frac{e^2}{2cm_e c^2} \simeq 1.4 \times 10^{-13}\ \text{cm} \tag{3.12}$$

Introduction of a lower limit to the integral (3.11) generally makes the theory nonlocal and the interaction of fields becomes tremendously complicated. However, in our present theory, a free electron behaves as a point-like particle and its nonlocal character in the virtual state is described by the additional tildon field. With the potential (3.8), we obtain the total energy of the electric field

$$U = \frac{e^2}{8\pi}\int_0^{\infty}\left[-\frac{\partial}{\partial r}\frac{1}{r}(1 - e^{-r/l_0})\right]^2 4\pi r^2\, dr = \frac{e^2}{4l_0} \tag{3.13}$$

3.3. A New Version of the Dirac Equation. In the same way as the Dirac equation was derived, one can also extract a modified Dirac equation from the Bopp equation. Since the Bopp equation is a fourth-order differential equation, the modified Dirac equation will be quadratic. The resultant equation is

$$\frac{l_0}{\hbar}(\not{p} + \tilde{m}_+ c)(\not{p} - \tilde{m}_- c)\psi = 0 \tag{3.14}$$

which reduces to the usual Dirac equation $(\not{p} - m_0 c)\psi = 0$ for $l_0 \to 0$. In the coordinate space, it is given as

$$-l_0\left(+i\not{\partial} + \frac{\tilde{m}_+ c}{\hbar}\right)\left(-i\not{\partial} + \frac{\tilde{m}_- c}{\hbar}\right)\psi = 0 \tag{3.15}$$

Here, we used a usual notation, $\not{p} = \gamma_\mu p^\mu$, with the Dirac γ matrices which have the following properties,

$$\begin{aligned} \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu &= 2g_{\mu\nu} I \\ \gamma_0^2 = -\gamma_j^2 &= 1 \qquad (j = 1, 2, 3) \end{aligned} \tag{3.16}$$

When the conjugate operators are operated from the left of the modified Dirac equation (3.15), we can easily obtain the Bopp equation as follows:

$$\begin{aligned} 0 &= l_0^2\left(-i\not{\partial} + \frac{\tilde{m}_+ c}{\hbar}\right)\left(i\not{\partial} + \frac{\tilde{m}_- c}{\hbar}\right)\left(+i\not{\partial} + \frac{\tilde{m}_+ c}{\hbar}\right)\left(-i\not{\partial} + \frac{\tilde{m}_- c}{\hbar}\right) \\ &= l_0^2\left[\partial_\mu \partial^\mu + i\frac{(\tilde{m}_+ - \tilde{m}_-)c}{\hbar}\not{\partial} + \frac{\tilde{m}_+ \tilde{m}_- c^2}{\hbar^2}\right] \\ &\quad \times \left[\partial_\nu \partial^\nu - i\frac{(\tilde{m}_+ - \tilde{m}_-)c}{\hbar}\not{\partial} + \frac{\tilde{m}_+ \tilde{m}_- c^2}{\hbar^2}\right] \\ &= [l_0^2(\partial_\mu \partial^\mu)(\partial_\nu \partial^\nu) + (l_0 c/\hbar)^2(\tilde{m}_+^2 + \tilde{m}_-^2)\partial_\mu \partial^\mu + l_0(c/\hbar)^4 \tilde{m}_+^2 \tilde{m}_-^2]\psi \\ &= [(1 + l_0^2 \partial_\mu \partial^\mu)\partial_\nu \partial^\nu + (m_0 c/\hbar)^2]\psi \end{aligned}$$

Here, we used the facts that $\not{\partial}\not{\partial} = \partial_\mu \partial^\mu$, $\tilde{m}_+^2 + \tilde{m}_-^2 = (\hbar/l_0 c)^2$ and $\tilde{m}_+^2 \tilde{m}_-^2 = (m_0 \hbar/l_0 c)^2$.

The solution of equation (3.15) is the superposition of a Dirac (spinor) field and a Dirac (spinor)–tildon field. In a free space, only the Dirac (spinor) field can appear as a physical entity and the Dirac (spinor)–tildon field, which can appear only in the virtual processes, is suppressed by the condition (2.9). Thus, the motion of the Dirac particle in a free space is described by the equation

$$(\not{p} - \tilde{m}_- c)\psi = 0$$

whose solution satisfies the modified equation (3.14) as well.

3.4. Propagators. Since the equations of motion for a photon and a Dirac particle are given by (3.4) and (3.15), it is easy to obtain the propagators for these particles.

Let us, for example, consider the equation of motion for a photon:

$$(1 + l_0^2 \partial^\mu \partial_\mu) \partial^\nu \partial_\nu \psi(x) = -\rho(x) \tag{3.17}$$

where $\rho(x)$ is a source of the field. The Fourier expansions of $\psi(x)$ and $\rho(x)$ are

$$\psi(x) = \frac{1}{(2\pi)^4} \int \Delta(k) e^{ikx} \, d^4k$$
$$\rho(x) = \frac{1}{(2\pi)^4} \int \rho(k) e^{ikx} \, d^4k \tag{3.18}$$

Substituting (3.18) into (3.17), we obtain

$$(1 - l_0^2 k^2) k^2 \Delta(k) = \rho(k) \tag{3.19}$$

From (3.18) and (3.19), the solution $\psi(x)$ is expressed by $\rho(x)$ as

$$\psi(x) = \frac{1}{(2\pi)^4} \int \frac{\rho(k)}{(1 - l_0^2 k^2) k^2} e^{ikx} \, d^4k \tag{3.20}$$

Thus, the propagator of photon yields

$$\frac{1}{(1 - l_0^2 k^2) k^2} = \frac{1}{k^2} - \frac{1}{k^2 - (1/l_0)^2} \tag{3.21}$$

where the second term in the right-hand side is the propagator of the photon–tildon. The propagator obtained here has exactly the same form as that given by Feynman (1949), except that in his case $\Lambda = 1/l_0 \to \infty$ after calculation of the physical quantities. Now we define the quantities $\psi(k)$ and $\phi(\bar{k})$ as

$$k^2 \psi(k) = \rho(k)$$
$$\bar{k}^2 \phi(\bar{k}) = \bar{\rho}(\bar{k}) \tag{3.22}$$

where

$$k_0 = |\mathbf{k}|, \qquad \bar{\mathbf{k}} = \mathbf{k}, \qquad \bar{k}_0^2 = \mathbf{k}^2 + 1/l_0^2, \qquad \text{and} \qquad \frac{\rho(k)}{k_0} e^{ikx} = \frac{\bar{\rho}(\bar{k})}{k_0} e^{ikx}$$

Since $k_0 \, d^4k = \bar{k}_0 \, d^4\bar{k}$, the solution (3.20) can be rewritten as

$$\begin{aligned}
\psi(x) &= \frac{1}{(2\pi)^4} \int \left[\frac{1}{k^2} - \frac{1}{k^2 - (1/l_0)^2}\right] \rho(k) e^{ikx} \, d^4k \\
&= \frac{1}{(2\pi)^4} \int \frac{\rho(k)}{k^2} e^{ikx} \, d^4k + \frac{1}{(2\pi)^4} \int \frac{\bar{\rho}(\bar{k})}{\bar{k}^2} \frac{k_0}{\bar{k}_0} e^{i\bar{k}x} \, d^4k \\
&= \frac{1}{(2\pi)^4} \int \psi(k) e^{ikx} \, d^4k + \frac{1}{(2\pi)^4} \int \phi(\bar{k}) e^{ikx} \, d^4\bar{k}
\end{aligned} \tag{3.23}$$

Here, the first term on the right-hand side corresponds to a photon while the second term expresses a photon–tildon field. This is exactly the same result as that obtained in (3.6). Similarly, the propagator for the Dirac particle is given by

$$\frac{1}{(l_0/\hbar)(p\!\!\!/ + \tilde{m}_+ c)(p\!\!\!/ - \tilde{m}_- c)} = \frac{(\hbar/l_0 c)}{(\tilde{m}_+ + \tilde{m}_-)}\left[\frac{1}{p\!\!\!/ - \tilde{m}_- c} - \frac{1}{p\!\!\!/ + \tilde{m}_+ c}\right] \quad (3.24)$$

where the second term on the right-hand side is also the propagator of the tildon.

4. INDEFINITE METRIC AND *S* MATRIX

In this section we shall give a brief explanation of the indefinite metric which our basic equation of motion contains and a discussion on unitarity of the *S* matrix.

It is generally said that if the interaction is Hermitian the *S* matrix satisfies the unitary condition and the propagator has a positive sign and obeys a positive definite metric. While if the interaction is anti-Hermitian, unitarity of the *S* matrix breaks down and the propagator has a negative sign and obeys a negative metric.

Introducing an indefinite metric induces a pseudounitary condition for the *S* matrix, which seems to be necessary and sufficient as a condition imposed on the *S* matrix, when the physical quantities such as expectation values and norms are redefined. The only problem is how to handle the appearance of a negative probability. Our present theory may be successful in dealing with these problems.

4.1. Indefinite Metric and Norm. As was seen in the previous section, the state is generally given by the superposition of the particle state $|\phi_i\rangle$ and the tildon state $|\phi_i\rangle\rangle$, i.e.,

$$\|\phi_i\rangle = |\phi_i\rangle + |\phi_i\rangle\rangle$$

and $|\phi_i\rangle$ and $|\phi_i\rangle\rangle$ are neither coupled nor overlapped. If the interaction H is not Hermitian but pseudo-Hermitian, i.e.,

$$\eta H^+ \eta = H, \qquad \eta^2 = 1, \qquad \eta^+ = \eta \quad (4.1)$$

the wave function is normalized as

$$\begin{aligned}\langle\phi_i|\eta|\phi_j\rangle &= {}^N R_i \delta_{ij} \\ \langle\langle\phi_i|\eta|\phi_j\rangle\rangle &= {}^A R_i \delta_{ij} \\ \langle\phi_i\|\eta\|\phi_j\rangle\rangle &= ({}^N R_i + {}^A R_j)\delta_{ij} = R_i \delta_{ij}\end{aligned} \quad (4.2)$$

where ${}^{N}R_i^2 = {}^{A}R_i^2 = R_i^2 = 1$ and then the expectation value of a quantity A is given by

$$A_{ij} = \frac{1}{R_i} \langle \phi_i \| \eta A \| \phi_j \rangle \tag{4.3}$$

It seems to be convenient to take ${}^{N}R_i = 1$ and ${}^{A}R_i = (-1)^i$. Here, it should not be taken that $R_i = {}^{N}R_i + {}^{A}R_i = 1 + (-1)^i = 2$ or 0 for $i =$ even or odd, because the particle state is never coupled to the tildon state and *the norms of the particle and the tildon are independent of each other.* The tildon is actually not a real particle but only a reflection of the discrete space-time. Thus, we have the indefinite metric

$$\langle \phi_n | \eta | \phi_m \rangle = \delta_{nm} \qquad \text{for normal states}$$

$$\langle\langle \phi_n | \eta | \phi_m \rangle\rangle = (-1)^n \delta_{nm} \qquad \text{for abnormal states} \tag{4.4}$$

which are, respectively, associated with the Hermitian and anti-Hermitian interaction, i.e., $H^+ = H$ and $H^+ = -H$. In our case, the tildon corresponding to the abnormal states is not observable for the sake of the relation (2.9), while the particle corresponds to the normal states which have a positive definite metric.

4.2. *S* Matrix. Now, defining the S matrix in the conventional way as

$$\|\psi\rangle = S \|\phi\rangle \tag{4.5}$$

where $\|\phi\rangle$ and $\|\psi\rangle$ are the initial and final states, respectively, we find that

$$\langle \psi \| \eta \| \psi \rangle = \langle \phi \| S^+ \eta S \| \phi \rangle \tag{4.6}$$

In order to have the norm,

$$\langle \psi \| \eta \| \psi \rangle = \langle \phi \| \eta \| \phi \rangle \tag{4.7}$$

one should have the relation

$$S^+ \eta S = \eta \tag{4.8}$$

This is the condition that we should have in the present theory instead of the unitary condition in the usual theory. Let us call it the "pseudounitary condition" hereafter. With the conventional expression of the S matrix,

$$S = 1 - \frac{i}{\hbar} \int_{-\infty}^{\infty} H(t)\, dt - \frac{1}{\hbar^2} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 H(t_1) H(t_2) + \cdots \tag{4.9}$$

the pseudounitary condition can also be obtained as

$$\begin{aligned} S^+\eta S = \eta &+ \frac{i}{\hbar}\int_{-\infty}^{\infty} dt[H^+(t)\eta - \eta H(t)] \\ &- \frac{1}{2\hbar^2}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\{H^+(t_1)[H^+(t_2)\eta - \eta H(t_2)] \\ &\qquad - [H^+(t_1)\eta - \eta H(t_1)]H(t_2)\} + \cdots \\ = \eta & \end{aligned}$$

where we used the relation (4.1). Thus, the following is clarified. If the interaction is pseudo-Hermitian, the S matrix is pseudounitary and the norm is given by (4.7). On the other hand, if the norm is given as (4.7) or (4.4) with the indefinite metric η, the S matrix is pseudounitary and the interaction is not Hermitian but pseudo-Hermitian. If the states are not associated with the tildon field but with the particle field, i.e., normal states, (4.7) is automatically reduced to the usual unitary condition. The reason why the indefinite metric has not been accepted so far is that the negative probability appeared in the physical processes and accordingly the pseudounitary condition was unsatisfactory in the usual theory. However, if only the negative probability is well handled, the indefinite metric may save the theory. In our theory, the tildon is allowed to appear only in virtual states by the mass relation (2.9) and, therefore, pseudounitarity is necessary and sufficient as a condition imposed on the S matrix.

It can be seen from equations (3.1) and (3.6) that the tildon is always associated with the particle and it cannot exist alone because it is just a reflection of the discrete space-time and makes a significant role only when the particle is put in the space-time. Therefore, the number of the particle cannot be smaller than that of the tildon in any processes.

4.3. Probability. In the frame of our present theory, the probability is expressed as

$$P_n = \langle\psi\|\phi_n\rangle\eta\langle\phi_n\|\psi\rangle \tag{4.10}$$

while it is given by $|\langle\phi_n\|\psi\rangle|^2$ in the usual theory. Of course, for the normal state, (4.10) is reduced to the usual expression. For the abnormal state, we find that

$$P_n = \langle\psi\|\phi_n\rangle\rangle(-1)^n\langle\langle\phi_n\|\psi\rangle = (-1)^n|\langle\langle\phi_n\|\psi\rangle|^2 \tag{4.11}$$

which is negative when n is odd. This means that we have negative probability for the state in which an odd number of tildons exists. This fact is no longer a problem in our case because the tildon can exist only in the virtual state for the sake of the mass relation (2.9). Though the probability is positive for even number of the tildon, they are also confined in the virtual state by the mass relation.

Let us set up an axiom as follows: the probability is expressed by negative values for the state where an odd number of the tildons exists and by positive values for the state where an even number of the tildons exists.

With this axiom, we do not have any difficulty in our present theory. Introducing a quantity ϵ (where $\epsilon^2 = 1$), which is given as $\epsilon^+ = \epsilon$ and $\epsilon^+ = -\epsilon$ associated, respectively, with the Hermitian and anti-Hermitian interactions, $H^+ = H$ and $H^+ = -H$, we give the probability density

$$|\epsilon^n \phi_1 \cdots \phi_n|^2 = (\epsilon^+)^n (\epsilon)^n |\phi_1 \cdots \phi_n|^2$$

$$= \begin{cases} (-1)^n \epsilon^{2n} |\phi_1 \cdots \phi_n|^2 = (-1)^n |\phi_1 \cdots \phi_n|^2 & \text{for } \epsilon^+ = -\epsilon \\ \epsilon^{2n} |\phi_1 \cdots \phi_n|^2 = |\phi_1 \cdots \phi_n|^2 & \text{for } \epsilon^+ = \epsilon \end{cases}$$

The quantity ϵ is interpreted as a representation of the particle and the tildon.

Now summing the probability (4.10) over n, we obtain the probability in the whole space

$$\sum_n P_n = \sum_n \langle \psi \| \phi_n \rangle \eta \langle \phi_n \| \psi \rangle \tag{4.12}$$

$$= \sum_n \langle \phi \| S^+ \| \phi_n \rangle \eta \langle \phi_n \| S \| \psi \rangle \tag{4.13}$$

If we have a relation

$$\sum_n \| \phi_n \rangle \eta \langle \phi_n \| = \eta \tag{4.14}$$

we find

$$\sum_n P_n = \langle \psi \| \eta \| \psi \rangle \tag{4.15}$$

by (4.12) and

$$\sum_n P_n = \langle \phi \| S^+ \eta S \| \phi \rangle = \langle \phi \| \eta \| \phi \rangle \tag{4.16}$$

by (4.13) and (4.8). These results are consistent with (4.7). Accordingly, it can be said that when the norm is given by (4.7), the relation (4.14) holds. For the normal states, the relation (4.14) reduces to the usual closure relation, i.e., $\sum_n |\phi_n\rangle\langle\phi_n| = 1$. It can also be rewritten

$$\sum_n \eta \| \phi_n \rangle \langle \phi_n \| \eta = 1 \tag{4.14'}$$

4.4. Matrix Element. In calculation of the matrix elements, the first-order term is obtained by making use of (4.2), (4.3), (4.4), and (4.14′) as

$$\langle \phi_f \| H \| \phi_i \rangle = \sum_l \langle \phi_f \| \eta \| \phi_l \rangle \langle \phi_l \| \eta H \| \phi_i \rangle$$

$$= \sum_l R_l \delta_{fl} \langle \phi_l \| \eta H \| \phi_i \rangle = \sum_l R_l^2 \delta_{fl} H_{li} = H_{fi} \tag{4.17}$$

The numerator of the second-order term (i.e., apart from the energy denominator) is calculated as follows:

$$\langle \phi_f \| HH \| \phi_i \rangle = \sum_l \langle \phi_f \| H\eta \| \phi_l \rangle \langle \phi_l \| \eta H \| \phi_i \rangle$$

which can be written by help of the relation (4.1)

$$\eta H = H\eta \qquad \text{for } H^+ = H \text{ (normal)}$$

$$\eta H = -H\eta \qquad \text{for } H^+ = -H \text{ (abnormal)}$$

as

$$\langle \phi_f \| HH \| \phi_i \rangle = \begin{cases} \sum_l \langle \phi_f \| \eta H | \phi_l \rangle \langle \phi_l | \eta H \| \phi_i \rangle = \sum_l H_{fl} H_{li} & \text{for } H^+ = H \\ -\sum_l \langle \phi_f \| \eta H | \phi_l \rangle \rangle \langle \langle \phi_l | \eta H \| \phi_i \rangle = -\sum_l H_{fl} H_{li} & \\ & \text{for } H^+ = -H \end{cases} \tag{4.18}$$

Accordingly, for the case in which the tildon appears in the intermediate states the matrix element has a negative sign, i.e., the tildon propagator has a negative sign while the particle propagator has a positive sign. As has been seen herein, the negative sign of the tildon propagator is due to the indefinite metric. The fact mentioned above can be expressed as follows: the negative sign of the tildon propagator indicates that the tildons are coupled to the charges by a factor $-e^2$, so that the interaction Lagrangian would have the coupling constant, which implies that H is not Hermitian.

A discussion on the indefinite metric was also given by Lee and Wick (1969) from a different point of view.

5. CONCLUDING REMARKS

Under the hypothesis that a fundamental length exists of this nature, the basic equations of motion which are Lorentz invariant have been derived. Assuming that the particle mass is a function of l_0, we have found the very important mass relation which is the condition for confirment of the tildon field with indefinite metric in the virtual state. This mass relation indicates that the particle mass larger than $(\hbar/2l_0c)$ cannot exist. It can be implicated to the concept of compensation as follows: The constants c and h appearing in the special theory of relativity and quantum theory represent the departure of these theories from classical physics. The constant c compensates for loss of the invariance of distance and time interval, while the constant h compensates for the impossibility of simultaneous measurement of the position and momentum of a particle with arbitrary precision. The constant

c also imposes an upper limit upon the velocity of the particle. The third universal constant l_0 introduced in our theory compensates for loss of the rest mass of the elementary particle larger than $(\hbar/2l_0c)$ and it imposes a minimum upon the measurable length.

There is another idea for removal of divergences. Namely, a form factor is taken into account at vertexes (Ringhofer and Salecker, 1975; Rich and Wesley, 1972). The propagator of the photon in our theory can be written as

$$\frac{1}{k^2} - \frac{1}{k^2 - (1/l_0^2)} = \frac{1}{k^2}\left[1 - \frac{k^2}{k^2 - (1/l_0^2)}\right] \equiv \frac{1}{k^2} F(k^2, l_0^2)$$

where $F(k^2, l_0^2)$ takes the place of the form factor and it becomes a pole type

$$(1 + l_0^2k^2) \text{ for } k^2 < 1/l_0^2$$

The value of the universal constant l_0 can be determined only by experiment. The recent experiments by the Stanford group on $e^+e^- \to e^+e^-$, $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to \gamma\gamma$ confirmed that the QED is valid up to 4×10^{-15} cm (Hofstadter, 1975). With these data, we can find $l_0 < 4 \times 10^{-15}$ cm. Since the very accurate measurement of the anomalous magnetic moments for the electron and the muon are now available (Rich and Wesley, 1972; Walls and Stein, 1973; Bailey et al., 1975), we can also estimate l_0 with these data. When the vertex correction is calculated by using our propagators, we obtain an additional term to the Schwinger term of the magnetic moment. Comparing the theoretical value with the recent data (Rich and Wesley, 1972; Bailey et al., 1975), we obtain $l_0 \simeq 10^{-16}$ cm for a photon and $l_0 \simeq 10^{-20}$ cm for a fermion. A detailed discussion will be given in a separate paper. Kirzhnits and Chechin (1968) investigated the radiation from a charged particle on the basis of quantized space-time, and by comparing the theoretical width with that manifested in the Mössbauer effect they estimated an upper limit of the fundamental length, $l_0 \simeq 10^{-20}$ cm.

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APPENDIX

When a particle (say, a proton, its mass m_p) is put in a certain field, the particle will fluctuate owing to the field fluctuation, and, thus, it cannot be localized in a sphere whose radius is smaller than a certain value of R. Although the position of the free particle is quantum-mechanically uncertain by $\Delta x > (\hbar/m_pc)$, we shall here estimate the uncertainty of the particle position due to the field fluctuation.

Let us consider the Lagrangian with a nonrelativistic potential ϕ,

$$L = -m_p c^2 + \frac{m_p v^2}{2} - \phi \tag{A.1}$$

where v is the velocity of the particle with the mass m_p. The action is given by

$$\begin{aligned} S &\equiv -m_p c \int ds \\ &= \int L\, dt = -m_p c \int \left(c - \frac{v^2}{2c} + \frac{\phi}{m_p c}\right) dt \end{aligned} \tag{A.2}$$

and then, the world-line is

$$ds = c\, dt - \frac{1}{2c} \mathbf{v} \cdot d\mathbf{r} + \frac{\phi}{m_p c}\, dt \tag{A.3}$$

Its square yields

$$(ds)^2 = c^2\left(1 + \frac{2\phi}{m_p c^2}\right)(dt)^2 - (d\mathbf{r})^2 \tag{A.4}$$

where the terms of order $(v/c)^2$ are neglected. Here, we have a condition,

$$g_{00} \equiv 1 + \frac{2\phi}{m_p c^2} \geqslant 0 \tag{A.5}$$

When the proton energy due to the fluctuation is expressed as the mass increase, the position of the proton is uncertain by

$$\Delta x (\equiv 2R) \equiv c \Delta t \gtrsim \frac{\hbar c}{\Delta E} = \frac{\hbar}{\Delta m_p c} \tag{A.6}$$

The gravitational potential due to the fluctuation between two protons at $r \gtrsim R$ is, then, given by

$$\phi(r) = -\frac{G(m_p + \Delta m_p)^2}{r} + \frac{G m_p{}^2}{r} \simeq -\frac{2G m_p \Delta m_p}{r} \tag{A.7}$$

Combining (A.5), (A.6), and (A.7), we obtain at $r = R$,

$$0 \leqslant g_{00} = 1 + \frac{2\phi(R)}{m_p c^2} \simeq 1 - \frac{4G\Delta m_p}{Rc^2} \lesssim 1 - \frac{2G\hbar}{R^2 c^3} \tag{A.8}$$

Thus, we find

$$R \gtrsim (2G\hbar/c^3)^{1/2} \tag{A.9}$$

By the discussion given above, it has turned out that the particle in the gravitational field cannot be localized in the sphere of radius less than $(2G\hbar/c^3)^{1/2}$. The discussion was initially given by Mead (1964).

TABLE 1. The values of $R_0 = (2F)^{1/2}(\hbar/m_pc)$

	F	R_0 (cm)
Gravitation	$Gm_p^2/\hbar c = 0.6 \times 10^{-38}$	2.3×10^{-33}
Weak int.[a]	$\tilde{g}_w^2/4\pi\hbar c = (g_w^2/4\pi\hbar c)(m_pc/\hbar)^2 = 0.25 \times 10^{-7}$	4.7×10^{-18}
E.M. int.	$e^2/\hbar c = 1/137$	2.5×10^{-15}
Strong int.	$g^2/4\pi\hbar c = 1$	3.0×10^{-14}

[a] For the decay $\mu^- \rightarrow e^- + \nu + \bar{\nu}$,

$$\frac{\tilde{g}_w^4}{4\pi(\hbar c)^2} = \frac{1}{4\pi}\left(\frac{g_w^2}{\hbar c}\right)^2\left(\frac{m_pc}{\hbar}\right)^4 = 0.76 \times 19^{-14}$$

$g_w^2 = 0.27 \times 10^{-44}$ cm^3 MeV.

Since the electric field is $-e^2/r$, replacing G by e^2/m_p^2 in (A.9) we obtain the localizability radius in the electric field

$$R \gtrsim (2\alpha)^{1/2}(\hbar/m_pc) \tag{A.10}$$

where $\alpha = (e^2/\hbar c)$ is the fine-structure constant.

The field associated with a scalar boson exchange is expressed by the Yukawa-type potential

$$-\frac{g^2}{4\pi}\frac{e^{-\mu cr/\hbar}}{r} \underset{r=R}{\rightarrow} \simeq -\frac{g^2}{4\pi R} \tag{A.11}$$

Replacing G by $g^2/4\pi m_p^2$, we find

$$R \gtrsim \left[2\left(\frac{g^2}{4\pi\hbar c}\right)\right]^{1/2}\left(\frac{\hbar}{m_pc}\right) \tag{A.12}$$

Thus, the general form of the localizability conditions for the proton is

$$R \gtrsim (2F)^{1/2}\left(\frac{\hbar}{m_pc}\right) \tag{A.13}$$

where F is the coupling constant. The region where a particle can be localized in the field depends on the strength of the coupling field, and a particle with a mass m cannot be bound in the sphere whose radius is less than $(2F)^{1/2} \times (\hbar/mc)$. Table 1 shows the minimum radius of localization for various coupling constants. It is consistent that the localizability becomes better as the coupling constant becomes weaker.

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